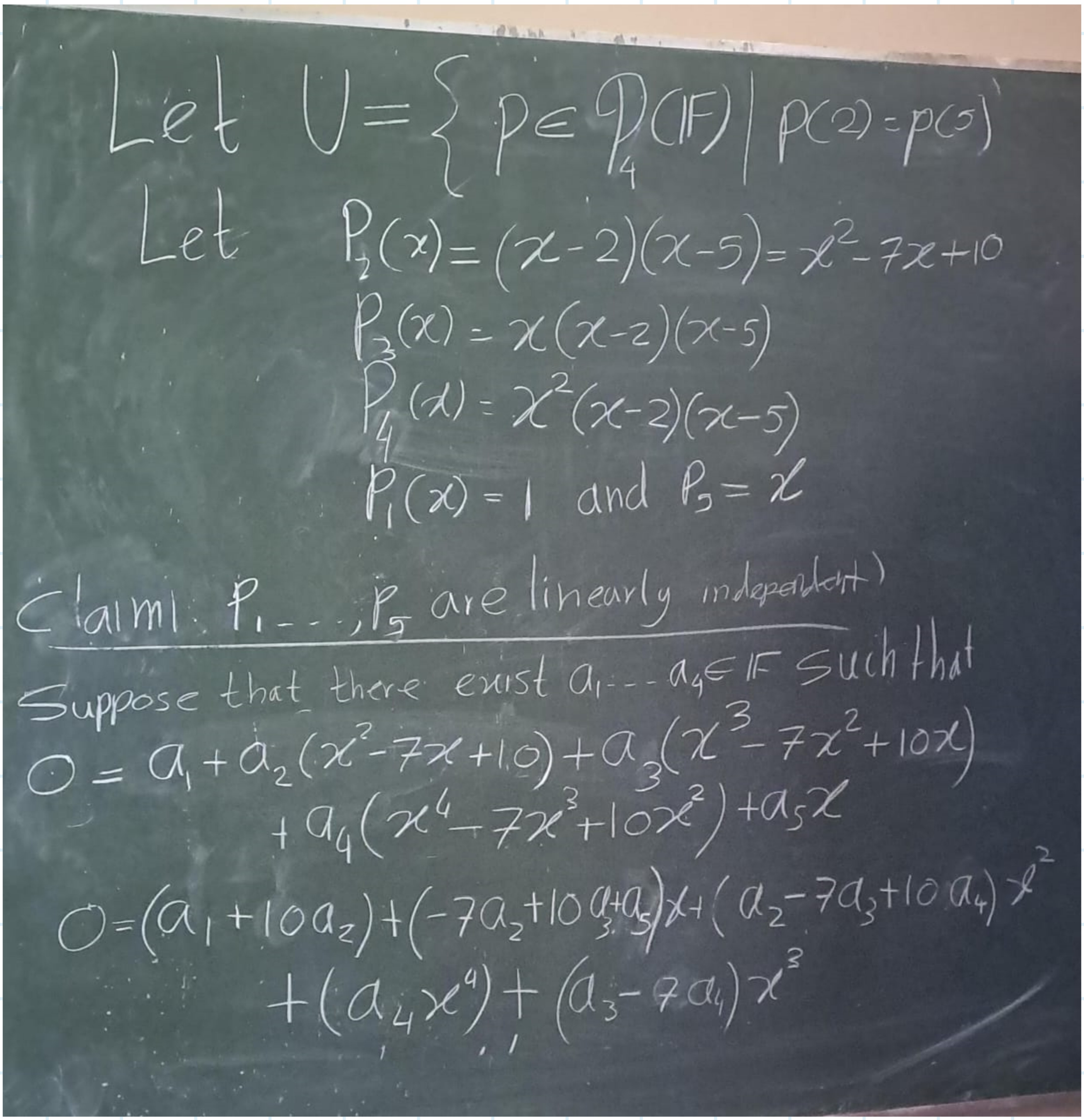
5 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.



Let $U = \{ p \in \mathcal{P}_4(\mathbb{F}) \mid p(2) = p(5))$

Let $P_2(x) = (x-2)(x-5) = x^2 - 7x + 10$

$P_3(x) = x(x-2)(x-5)$

$P_4(x) = x^2(x-2)(x-5)$

$P_1(x) = 1$ and $P_5 = x$

Claim: $P_1, \dots, P_5$ are linearly independent)

Suppose that there exist $a_1 \dots a_4 \in \mathbb{F}$ such that

$$0 = a_1 + a_2(x^2 - 7x + 10) + a_3(x^3 - 7x^2 + 10x) + a_4(x^4 - 7x^3 + 10x^2) + a_5 x$$

$$0 = (a_1 + 10a_2) + (-7a_2 + 10a_3 + a_5)x + (a_2 - 7a_3 + 10a_4)x^2 + (a_4 x^4) + (a_3 - 7a_4)x^3$$

By comparing relevent terms,

- $\boxed{a_4=0}$
- $a_3-7a_4=0 \Rightarrow \boxed{a_3=0}$
- $a_2-7a_3+10a_4=0 \Rightarrow \boxed{a_2=0}$
- $-7a_2+10a_3+a_5=0 \Rightarrow \boxed{a_5=0}$
- $a_1+10a_2=0 \Rightarrow \boxed{a_1=0}$

Thus $a_1=a_2=a_3=a_4=a_5=0$

Therfore, $P_1,\dots,P_5$ are linearly independent.

Claim2: $P_1, \dots, P_5$ spans U.

Let $g(x) = k_0 + k_1 x^1 + k_2 x^2 + k_3 x^3 + k_4 x^4 \in U$.

$$g(x) = k_4(x^4 - 7x^3 + 10x^2) + (7k_4 + k_3)(x^3 - 7x^2 + 10x)$$
$$+ (39k_4 + 7k_3 + k_2)(x^2 - 7x + 10)$$
$$+ (203k_4 + 39k_3 + 7k_2 + k_1)x + k_0$$

Then,

$$g(x) = k_4 P_4(x) + (7k_4 + k_3)P_3(x) + (39k_4 + 7k_3 + k_2)P_2(x)$$
$$+ (203k_4 + 39k_3 + 7k_2 + k_1)P_5(x) + k_0 P_1(x)$$

Therefore, $P_1, \dots, P_5$ spans $\mathcal{P}_4(\mathbb{F})$

Hence, $P_1, \dots, P_5$ is a basis for $\mathcal{P}_4(\mathbb{F})$

Let $W = \text{span}\{x\} = \{kx \mid k \in \mathbb{R}\}$

Then, $U + W = \mathcal{P}_4(\mathbb{F})$ (by part b))

Furtur, we can observe that $U \cap W = \{0\}$

Therfore, $U \oplus W = \mathcal{P}_4(\mathbb{F})$

6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(2) = p(5) = p(6)\}$

$$P_3(x) = (x-2)(x-5)(x-6) \in U$$
$$P_4(x) = x(x-2)(x-5)(x-6) \in U$$

Claim 1: P_3 & P_4 are linearly independent

Suppose that $a, b \in \mathbb{F}$ such that

$$0 = aP_3(x) + bP_4(x) = a(x^4 - 13x^3 + 52x^2 - 60x) + b(x^3 - 13x^2 + 52x - 60).$$
$$0 = ax^4 + (-13a+b)x^3 + (52a-13b)x^2 + (-60a+52b)x + 60b$$

This implies, $a = b = 0$.

claim 2: $\cancel{P_0, P_1, P_2}$ $P_3\ P_4$ spans U.

Let $f(x) = kx^4 + lx^3 + mx^2 + nx + q$

$$f(x) = k(x^4 - 13x^3 + 52x^2 - 60x)$$
$$+ (l + 13k)(x^3 - 13x^2 + 52x - 60)$$
$$+ (m + 117k + 13l)x^2 + (n - 616k - 52l)x$$
$$+ (q + 60l + 780k)$$

$0 = f(2) = 16k + 8l + 4m + 2n + q$ —— (1)

$0 = f(5) = 625k + 125l + 25m + 5n + q$ —— (2)

$0 = f(6) = 1296k + 216l + 36m + 6n + q$ —— (3)

(1) - (2) $0 = -609k - 117l - 21m - 3n$ —— (4)

$0 = 671k + 91l + 11m + n$ —— (5)

$\left((4) + 3 \times (5)\right) \times \frac{1}{12}$, $0 = m + 117k + 13l$ —— (*)

$$\left(4 \times 11 + 5 \times 21\right) \times \frac{(-1)}{12}$$

$$(n - 616k - 52\ell) = 0 \quad \text{——} \quad ** $$

$$\left(1 \times 5 - 2 \times 2\right) \times \frac{1}{3}$$

$$q + 780k + 60\ell = 0 \quad \text{——} \quad ***$$

By $*$, $**$, $***$

$$f(x) = k(x^4 - 13x^3 + 52x^2 - 60x) + (\ell + 13k)(x^3 - 13x^2 + 52x - 60)$$

$$= kP_4 + (\ell + 13k)P_3$$

b) ~~W~~

~~$1, x, x^2$,~~

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = x^2$

$P_3(x) = (x-2)(x-5)(x-6)$

$P_4(x) = x(x-2)(x-5)(x-6)$

Claim: $P_0(x), \ldots, P_4(x)$ are linearly independent

Suppose that there exist $a_0, \ldots, a_4 \in \mathbb{F}$ such that

$$a_0 P_0(x) + \cdots + a_4 P_4(x) = 0$$

$$a_0 + a_1 x + a_2 x^2 + a_3(x^3 - 13x^2 + 52x - 60) + a_4(x^4 - 13x^3 + 52x^2 - 60x)$$

~~$(a_0 + a_3)$~~

$$(a_0 - 60a_3) + (a_1 + a_3(52) - 60(a_4))x + (a_2 - 13a_3 + 52a_4)x^2 + (a_3 - 13a_4)x^3 + a_4 x^4 = 0$$

Since $1, x, x^2, x^3, x^4$ are linearly independe

$a_4 = 0$

$(a_3 - 13a_4) = 0 \Rightarrow a_3 = 0$

$a_2 - 13a_3 + 52a_4 = 0 \Rightarrow a_2 = 0$

$a_1 + 52a_3 - 60a_4 = 0$

$\Rightarrow a_1 = 0$

$a_0 - 60a_3 = 0 \Rightarrow a_1 = 0$

Thus, $P_0(x), \ldots, P_4(x)$ is linearly independent.

Claim2: $P_0(x), \ldots, P_4(x)$ is a basis for $\mathcal{P}_4(\mathbb{F})$

~~By~~ We know that $\dim(\mathcal{P}_4(\mathbb{F})) = 5$

Then by 2.38 $P_0, \ldots, P_4$ are basis of $\mathcal{P}_4(\mathbb{F})$

c) ~~$W = \{1, x, x^2\}$~~

$W = \text{span}(1, x, x^2)$

$= \{k_0 + k_1 x + k_2 x^2 \mid k_0, k_1, k_2 \in \mathbb{F}\}$

Then $U + W = \mathcal{P}_4(\mathbb{F})$ (by part(b))

Further $U \cap W = \cancel{\emptyset}\ \{0\}$. Therefore, $U \oplus W = \mathcal{P}_4(\mathbb{F})$